

# A generalised Measurement Equation and van Cittert-Zernike theorem for wide-field radio astronomical interferometry

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## ABSTRACT

We derive a generalised van Cittert-Zernike (vC-Z) theorem for radio astronomy that is valid for partially polarized sources over an arbitrarily wide field-of-view (FoV). The classical vC-Z theorem is the theoretical foundation of radio astronomical interferometry, and its application is the basis of interferometric imaging. Existing generalised vC-Z theorems in radio astronomy assume, however, either paraxiality (narrow FoV) or scalar (unpolarized) sources. Our theorem uses neither of these assumptions, which are seldom fulfilled in practice in radio astronomy, and treats the full electromagnetic field. To handle wide, partially polarized fields, we extend the two-dimensional electric field (Jones vector) formalism of the standard “Measurement Equation” of radio astronomical interferometry to the full three-dimensional formalism developed in optical coherence theory. The resulting vC-Z theorem enables all-sky imaging in a single telescope pointing, and imaging using not only standard dual-polarized interferometers (that measure 2-D electric fields), but also electric tripoles and electromagnetic vector-sensor interferometers. We show that the standard 2-D Measurement Equation is easily obtained from our formalism in the case of dual-polarized antenna element interferometers. We find, however, that such dual-polarized interferometers can have polarimetric aberrations at the edges of the FoV that are often correctable. Our theorem is particularly relevant to proposed and recently developed wide FoV interferometers such as LOFAR and SKA, for which direction-dependent effects will be important.

**Key words:** telescopes; techniques: interferometric; techniques: polarimetric; instrumentation: interferometers; instrumentation: polarimeters.

## 1 INTRODUCTION

Polarimetric wide-field imaging is a recent and important trend in radio astronomy. Many new and planned radio telescopes such as LOFAR, LWA, MWA, and SKA have a wide field-of-view (FoV) as a major feature. Technologically, this has been made possible by the development of phased dipole arrays and focal plane arrays, both of which have an inherently wider FoV than traditional single-pixel radio telescopes. The motivation for wide FoV polarimetric radio telescopes in astronomy is that they facilitate the study of polarized phenomena not restricted to narrow fields such as the large, highly-structured, polarized features discovered in recent polarimetric galactic surveys (Taylor et al. 2006). With LOFAR now producing its first *all-sky* images (LOFAR Team 2007), a new era of polarimetric wide-field imaging is starting in radio astronomy.

Despite this trend, a complete theory for wide-field, polarimetric astronomical interferometry is lacking. Ultimately, classical interferometry is based on the far-zone form of the van Cittert-Zernike (vC-Z) theorem (Thompson et al. 2001; Mandel & Wolf

1995) which, in its original form, is a scalar theory<sup>1</sup> and only valid for narrow fields. Such restrictions are obviously unacceptable in astronomy where polarization often provides crucial astronomical information, and sources are distributed on the celestial sphere, not necessarily limited to small patches. In radio astronomy, a polarimetric, or vector (i.e. non-scalar), extension of the narrow-field vC-Z theorem was given by Morris et al. (1964), and more recently this was given a consistent mathematical foundation by Hamaker (2000) in the form of the so-called *Measurement Equation* (M.E.) of radio astronomy. On the other hand, a wide-field extension of the scalar vC-Z theorem for radio astronomy was given by Brouw (1971), and more recently, imaging techniques for scalar, wide-fields have been an active field of research (Cornwell & Perley 1992; Sault et al. 1999; Cornwell et al. 2005; McConnell et al. 2006). However, a generalised vC-Z theorem for radio astronomy that is both polarimetric and wide-field has not been derived. This may be because it has been assumed that such a theorem would be a trivial vector- or matrix-valued analogue of the

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<sup>1</sup> By scalar theory we mean a theory which only considers unpolarized radiation.

wide-field, scalar theory (Thompson et al. 2001). As we will see, however, the final result is not that simple.

The purpose of the present work is therefore to derive a vC-Z type relation by generalising the standard M.E. formalism to allow for arbitrarily wide fields. This is achieved by generalising the two-component Jones formalism to a three-component Wolf formalism (Wolf 1954). In what follows, we will derive a vC-Z relation that is valid for the entire celestial sphere and is fully polarimetric. We will show that the standard M.E. can be recovered through a two-dimensional projection. We will also show that a dual-polarized interferometer of short electric dipoles (Hertzian dipoles) is inherently aberrated polarimetrically. We also extend our vC-Z relation to include not only the full electric field, but also the full magnetic field, and thereby establish the complete second-order statistical description of the electromagnetic relation between source terms and interferometer response.

Our wide-field vC-Z relations should be of relevance to the subject of *direction-dependent effects* in radio interferometric imaging, which has attracted recent attention (Bhatnagar 2008).

## 2 DERIVING A vC-Z RELATION IN THREE-COMPONENT FORMALISM

The vC-Z theorem as used in astronomy is different from its original use in optical coherence. In astronomy one images sources on the celestial sphere based on localised measurements of their far-fields. This is possible because the vC-Z theorem provides an explicit relationship between the visibility measured directly by an interferometer and the brightness distribution of the sources (Thompson et al. 2001, chap 14). This makes the vC-Z relationship the foundation of synthesis mapping and interferometric imaging. Despite its importance, the vC-Z relations in use in astronomy are all either based on the paraxial approximation (narrow FoV) or they take the source emissions to be scalar. These simplifications are questionable when the sources cover wide fields or are highly polarized, as is often the case in radio astronomy.

Here we derive the full vector electromagnetic analogue to the wide-field, scalar vC-Z relation derived in Thompson et al. (2001, chap 14), and, as we will see, the final result is not simply a matrix- or vector-valued version of the scalar relation. We seek a relationship between the electromagnetic field coherence of a distribution of radio astronomical sources and the resulting electromagnetic field coherence at a radio astronomical interferometer. To simplify the discussion, we will use the term *interferometer* as a shorthand for radio astronomical interferometer<sup>2</sup>. The treatment is intended for Earth-based interferometer observations, but it also has space-based radio astronomical interferometry in mind.

Consider the problem illustrated in Fig. 1a). Radio emissions from far away sources, such as the point source  $S$ , are measured by an interferometer located in the domain  $\mathcal{M}$ . We want to establish a relationship between the coherence of the electric field emanating from a source distribution and the coherence of the electric field in  $\mathcal{M}$ .

Let us first consider a single point source  $S$  located at  $\mathbf{R}$ . The source is thus at a distance  $R = |\mathbf{R}|$  in the direction given by the unit vector (direction cosines)  $\mathbf{s} = \mathbf{R}/R$  with respect to the origin  $O$  approximately at the centre of  $\mathcal{M}$ . The source is within the

interferometers FoV  $\mathcal{F}$  which is centred on point  $C$  on the celestial sphere. The electric field from  $S$  is measured by the interferometer at pairs of positions  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in  $\mathcal{M}$ .  $\mathcal{M}$  is assumed to be bounded, so the maximum distance between any two measurement points  $D_M = \max_{\mathcal{M}} |\mathbf{r}_1 - \mathbf{r}_2|$  (maximum baseline) is finite. Although the source emission may be broadband we split it into narrow, quasi-monochromatic, spectral bands and consider a typical narrow (bandwidth much smaller than centre frequency) band centred on frequency  $\nu$ . The assumption that  $S$  is very far away, which quantitatively we take to mean that  $|\mathbf{R} - \mathbf{r}| \gg c/\nu$  for all  $\mathbf{r} \in \mathcal{M}$ , implies that the entire interferometer is in the far-field of the source. This means that the electric field  $\mathbf{E}$  at point  $\mathbf{r}_1$  at time  $t$  is

$$\mathbf{E}(\mathbf{r}_1, \mathbf{s}, t) = \mathcal{E} \left( \hat{\mathbf{R}}_1; \mathbf{s}, t - \frac{R_1}{c} \right) \frac{e^{-i2\pi\nu(t-R_1/c)}}{R_1} \quad (1)$$

where

$$R_1 = |\mathbf{R}_1| = |\mathbf{r}_1 - R\mathbf{s}|, \quad \hat{\mathbf{R}}_1 = \mathbf{R}_1/R_1 \quad (2)$$

and  $\mathcal{E} \in \mathbb{C}^3$  is the complex electric field amplitude vector emitted by the source at  $\mathbf{s}$  in the direction of  $\mathbf{r}_1$ .

Furthermore, in the far-field,  $\mathcal{E}$  is approximately transverse so that

$$\hat{\mathbf{R}}_1 \cdot \mathcal{E} \approx 0. \quad (3)$$

If we further assume that the angular extent of  $\mathcal{M}$  as seen from  $S$  is small, that is

$$|\mathbf{r}_1| \leq D_M \ll R, \quad (4)$$

then the interferometer is in the Fraunhofer far-field of the sources. In this case we can use the approximations  $\hat{\mathbf{R}}_1 \approx \mathbf{s}$  and  $1/R_1 \approx 1/R$  to simplify the expression for the electric field in  $\mathcal{M}$  to

$$\mathbf{E}(\mathbf{r}_1, \mathbf{s}, t) \approx \mathcal{E} \left( \mathbf{s}, t - \frac{R_1}{c} \right) \frac{e^{-i2\pi\nu(t-R_1/c)}}{R} \quad (5)$$

where

$$\mathbf{s} \cdot \mathcal{E} \approx 0. \quad (6)$$

Here we have dropped the first argument of  $\mathcal{E}$  since we are assuming that the angular variation of the sources emissions is small enough so that the  $\hat{\mathbf{R}}_1 \approx \mathbf{s}$  approximation implies that  $\mathcal{E}(\hat{\mathbf{R}}_1; \mathbf{s}, t) \approx \mathcal{E}(\mathbf{s}; \mathbf{s}, t)$ .

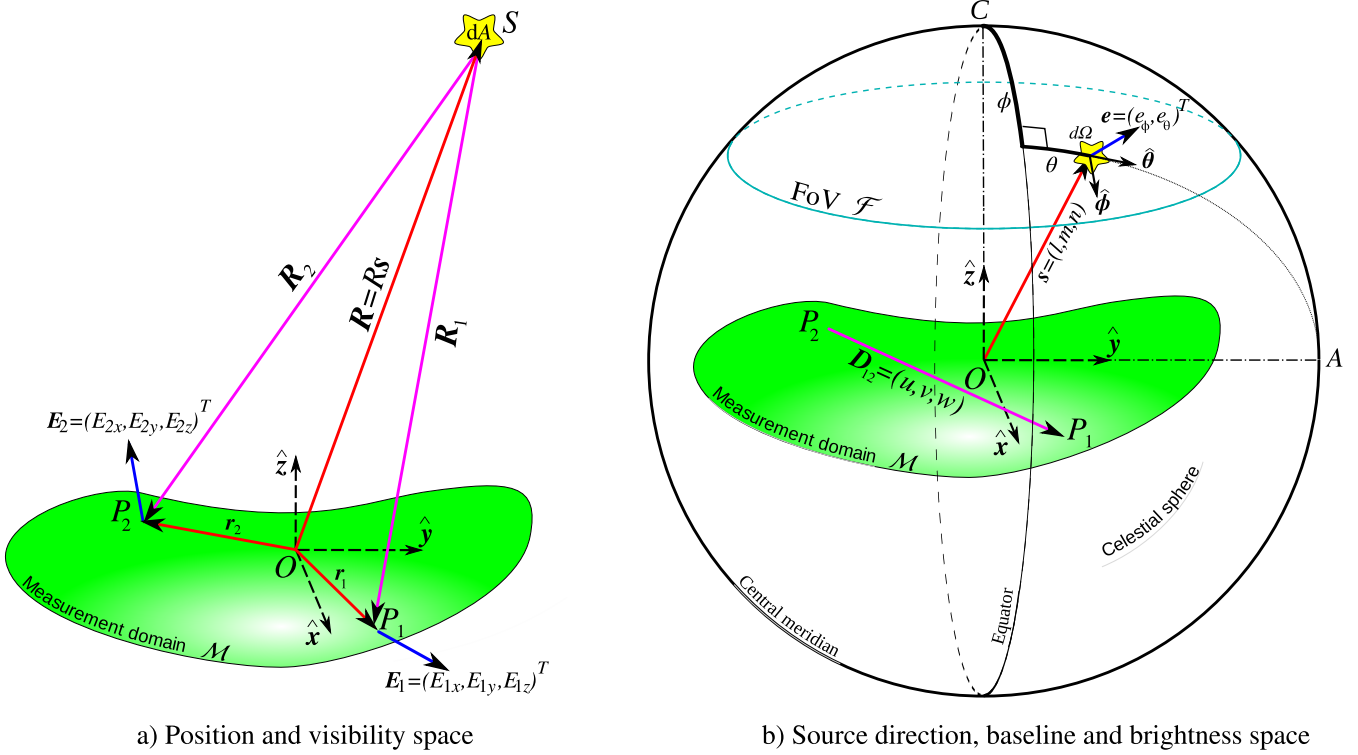
A similar expression to equation (5) for the field at  $\mathbf{r}_2$  is obtained by replacing  $\mathbf{r}_1$  with  $\mathbf{r}_2$  and  $R_1$  with  $R_2$ . The electric coherence matrix (tensor)  $\mathbf{\Gamma}$  can then be found by taking the outer product of the electric fields

$$\Gamma_{ij} = \langle E_i(\mathbf{r}_1, t) E_j^*(\mathbf{r}_2, t) \rangle \quad (7)$$

where  $E^*$  denotes complex conjugation,  $\langle \rangle$  denotes time averaging, and the subscripts  $i, j$  label Cartesian components  $x, y, z$ , which we will define more precisely later. We have written the electric coherence matrix, equation (7), as a  $3 \times 3$  complex matrix even though, for the single point source we are considering here, its rank is two and could therefore be expressed as a  $2 \times 2$  complex matrix. We keep the electric coherence matrix as a  $3 \times 3$  complex matrix since it is valid even when there are more than one point source.

We now move to the case of a finite number of point sources. The total electric field measured at points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  is now the sum of the fields from sources in directions  $\mathbf{s}_p$  for  $p = 1, 2, \dots$ . The

<sup>2</sup> Most of the formalism also applies to single-pixel radio telescopes as a special case when the interferometer baseline length is zero.



**Figure 1.** Coordinate systems and notations used in the text. a) A point source  $S$  located in look-direction  $\mathbf{s}$  at a distance  $R$  with cross-section  $dA$  emits radiation, and its electric field  $\mathbf{E}$  is measured at the pair of points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  located within the domain  $\mathcal{M}$ . At the approximate centre of  $\mathcal{M}$  is the origin of the coordinate systems  $O$ . b) We use both a spherical and a Cartesian system to express the wide-field vC-Z. These systems are constructed with reference to  $C$ , the centre of the FoV  $\mathcal{F}$ , and  $A$  an arbitrary direction orthogonal to  $C$  (which for Earth-based observations could be towards the North pole or zenith). The right-handed Cartesian system  $xyz$  is used for the source direction unit vector  $\mathbf{s} = l\hat{x} + m\hat{y} + n\hat{z}$  and the relative displacement vector  $\mathbf{D}_\lambda = u\hat{x} + v\hat{y} + w\hat{z}$ . The basis vector  $\hat{z}$  points towards  $C$  and  $\hat{y}$  points toward  $A$ . The angular part of the spherical coordinate system,  $\{\phi, \theta\}$ , has its zero point at  $C$ , and the pole  $A$  is at  $\theta = +\pi/2$ . Relative to an arbitrary point on the celestial sphere given by  $\mathbf{s}$ , the unit base vector  $\hat{\theta}$  is directed along the arc from the point towards  $A$ , and  $\hat{\phi}$  points in the anticlockwise direction as seen from  $A$ . These base vectors are used to specify the source electric field  $\mathbf{e} = e_\phi\hat{\phi} + e_\theta\hat{\theta}$  and consequently also the two-dimensional brightness matrix.

electric coherence matrix is therefore

$$\begin{aligned} \langle E_i(\mathbf{r}_1, t) E_j^*(\mathbf{r}_2, t) \rangle &= \sum_p \sum_q \langle E_i(\mathbf{r}_1, \mathbf{s}_p, t) E_j^*(\mathbf{r}_2, \mathbf{s}_q, t) \rangle \\ &\approx \sum_p \langle \mathcal{E}_i(\mathbf{s}_p, t) \mathcal{E}_j^*(\mathbf{s}_p, t) \rangle \frac{e^{-i2\pi\nu\mathbf{s} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/c}}{R^2} \end{aligned} \quad (8)$$

for  $i, j = x, y, z$ . In going from the double to the single sum we used the usual vC-Z assumption that the sources are spatially incoherent, that is, sources in different directions are statistically independent.

Until now we have considered only discrete sources. We can make the transition to the more general continuum source distribution by introducing the three-dimensional brightness matrix  $\mathbf{B}^{(3)}$  as a function of direction  $\mathbf{s}$  in a continuous source distribution

$$\sum_p \langle \mathcal{E}_i(\mathbf{s}_p, t) \mathcal{E}_j^*(\mathbf{s}_p, t) \rangle \rightarrow \int_{\text{source}} \mathbf{B}_{ij}^{(3)}(\mathbf{s}) dA, \quad (9)$$

where  $dA$  is the infinitesimal area of the source distribution. The superscript is to highlight the fact that  $\mathbf{B}^{(3)}$  is three-dimensional as opposed the usual two-dimensional brightness matrix (Hamaker 2000). The reason that the brightness matrix here is three-dimensional is simply because the full electric field amplitude  $\mathcal{E}$  is three-dimensional. However, due to equation (6), not all the com-

ponents of  $\mathbf{B}^{(3)}$  are arbitrary. In fact, we will show that it can be recast as one two-dimensional matrix.

By making the replacement (9) in equation (8), we obtain

$$\begin{aligned} \mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, 0) &= \int_{\text{source}} \mathbf{B}^{(3)}(\mathbf{s}) \frac{e^{-i2\pi\nu\mathbf{s} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/c}}{R^2} dA \\ &= \int_{\mathcal{F}} \mathbf{B}^{(3)}(\mathbf{s}) e^{-i2\pi\nu\mathbf{s} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/c} d\Omega, \end{aligned} \quad (10)$$

where  $d\Omega$  is an infinitesimal solid angle of the source distribution in  $\mathcal{F}$ . In this expression we see that the dependence on the pairs of position is only relative, that is, the coherence matrix depends only on  $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{D}$ , and so we recast the expression in terms of the vector

$$\mathbf{D}_\lambda = \frac{\nu}{c}(\mathbf{r}_1 - \mathbf{r}_2),$$

also known as the baseline vector measured in wavelengths ( $\lambda = c/\nu$ ).

In practice, rather than use  $\mathbf{\Gamma}$  directly, it is convenient and conventional to put the phase reference point at the centre of the FoV given by the direction  $\mathbf{s}_0$ . The result of the change in phase,

$$\mathbf{V}^{(3)}(\mathbf{D}_\lambda) = \exp(i2\pi\mathbf{s}_0 \cdot \mathbf{D}_\lambda) \mathbf{\Gamma}_\nu(\mathbf{r}_1, \mathbf{r}_2),$$

is the  $3 \times 3$  generalisation of the standard  $2 \times 2$  visibility matrix as defined by, for instance, Hamaker (2000).  $\mathbf{V}^{(3)}$  is a general complex matrix except for  $\mathbf{D}_\lambda = 0$  where it is Hermitian. It fulfill the

symmetry relation  $\mathbf{V}^{(3)}(-\mathbf{D}_\lambda) = \left(\mathbf{V}^{(3)}(\mathbf{D}_\lambda)\right)^\dagger$  where  $\dagger$  stands for Hermitian transpose.

If we use  $\mathbf{V}^{(3)}$  in equation (10) we arrive at

$$\mathbf{V}^{(3)}(\mathbf{D}_\lambda) = \int_{\mathcal{F}} \mathbf{B}^{(3)}(\mathbf{s}) e^{-i2\pi(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{D}_\lambda} d\Omega. \quad (11)$$

Equation (11) is a matrix version of the wide-field, scalar vC-Z relation and so includes the partial polarization of the (non-scalar) source distribution. However, it is not very useful in this form since it does not automatically fulfill the constraint (6). When applied to  $\mathbf{B}^{(3)}$ , this constraint becomes

$$\mathbf{B}^{(3)}\mathbf{s} = \mathbf{0} \quad (12)$$

for all directions  $\mathbf{s}$ , where  $\mathbf{s}$  is understood to be a column vector. The constraint can, however, easily be removed if we express  $\mathcal{E}$  in terms of spherical base vectors and set the radial component to zero. This leads us to introduce two coordinate systems: a Cartesian and a spherical.

We use the angular, or tangential, basis set  $\{\hat{\phi}, \hat{\theta}\}$  of a spherical polar coordinate system as the basis for the polarization of the transverse field of the source distribution on the celestial sphere, see Fig. 1b). To simplify the results, the spherical coordinate system is taken relative to the phase reference position of the interferometer assumed to be at  $C$ . In other words, the zero point of the spherical system,  $\{\phi, \theta\} = \{0, 0\}$ , (intersection of the equator and central meridian), is taken to coincide with  $C$ .  $\theta$  is the angle from the equator (positive in the hemisphere with the pole  $A$  and negative in the other hemisphere), and  $\phi$  is the position angle from the central meridian around  $AO$  in the anticlockwise sense looking along  $AO$ . In lieu of any other reference directions, the orientation of the spherical system around  $C$  is arbitrary, but for Earth-based measurements  $A$  could be directed towards the North pole or zenith.

One should note that  $\{\hat{\phi}, \hat{\theta}\}$  are consistent with Ludwig's second definition as detailed in Ludwig (1973) with the understanding that the antenna boresight in Ludwig (1973) is here at  $C$ , that Ludwig's reference polarization unit vector  $\hat{i}_{\text{ref}}$  is here  $\hat{\theta}$ , and Ludwig's cross polarization unit vector  $\hat{i}_{\text{cross}}$  is here  $\hat{\phi}$ . See (Piepmeier & Simon 2004) for the use of Ludwig's third definition in a vC-Z relation.

The Cartesian system, with base vectors  $\{\hat{x}, \hat{y}, \hat{z}\}$ , is defined so that  $\hat{z}$  is in the direction of  $C$ , and  $\hat{y}$  is in the direction of the pole  $A$ . In terms of the Cartesian system, we can explicitly write the components of the vectors in equation (11) as

$$\mathbf{s} = l\hat{x} + m\hat{y} + n\hat{z} = (l, m, n)^T, \quad (13)$$

$$\mathbf{s}_0 = \hat{z} = (0, 0, 1)^T, \quad (14)$$

$$\mathbf{D}_\lambda = u\hat{x} + v\hat{y} + w\hat{z} = (u, v, w)^T \quad (15)$$

where

$$n = \pm\sqrt{1-l^2-m^2} \quad (16)$$

and where the superscript  $T$  stands for vector transpose. All these vectors are unit vectors with real-valued components ranging between  $-1$  and  $+1$ . These definitions of the  $uvw$  and  $lmn$  spaces are the same as the usual definitions for Earth-based observations, see e.g., Thompson et al. (2001). Also the matrices in equation (11) are to be considered in what follows as being expressed in the Cartesian system.

The relationships between the spherical and Cartesian systems

base vectors are

$$\hat{\phi} = \frac{1}{\sqrt{1-m^2}} (n\hat{x} - l\hat{z}) \quad (17)$$

$$\hat{\theta} = \frac{1}{\sqrt{1-m^2}} (-lm\hat{x} + (1-m^2)\hat{y} - mn\hat{z}). \quad (18)$$

Using these spherical and Cartesian systems we can express the three-dimensional transverse electric field as

$$\mathcal{E} = \mathbf{T}\mathbf{e} \quad (19)$$

where

$$\mathbf{e}(l, m) = \begin{pmatrix} e_\phi(l, m) \\ e_\theta(l, m) \end{pmatrix}$$

is the Jones vector in spherical (rather than the usual Cartesian) components and

$$\mathbf{T} = \frac{1}{\sqrt{1-m^2}} \begin{pmatrix} n & -lm \\ 0 & 1-m^2 \\ -l & -mn \end{pmatrix} \quad \text{for } m^2 \neq 1 \quad (20)$$

is the  $3 \times 2$  transformation matrix between the components given by the equations (17) and (18). Note that this transformation is possible for all directions on the celestial sphere except for  $m = \pm 1$ , i.e., the poles of the spherical coordinate system.

If one wishes to use a polar spherical system in which the centre of the FoV is not on the equator, as it is here, but rather at some declination  $\Theta$ , then one simply replaces  $\mathbf{T}$  with  $\mathbf{T}'$  defined in appendix B, equation (B6). This assumes that the pole  $A$  is towards the Earth's North pole.

It is easy to show that

$$\mathbf{s}^T \mathbf{T}\mathbf{e} = 0,$$

so the transverse electric field expressed according to equation (19) does indeed fulfill equation (6). So if we use  $\mathbf{T}\mathbf{e}$  rather than  $\mathcal{E}$  in equation (11) we would have an unconstrained vC-Z equation. We can introduce this replacement by rewriting  $\mathbf{B}^{(3)}$  using equation (19), so

$$\mathbf{B}^{(3)} = \mathbf{T}\mathbf{B}\mathbf{T}^T \quad (21)$$

where, suppressing the dependence on  $(l, m)$ ,

$$\mathbf{B} = \begin{pmatrix} \langle |e_\phi|^2 \rangle & \langle e_\phi e_\theta^* \rangle \\ \langle e_\theta e_\phi^* \rangle & \langle |e_\theta|^2 \rangle \end{pmatrix} \quad (22)$$

is the  $2 \times 2$  brightness matrix, but in spherical rather than Cartesian coordinates. By this we mean that, for an arbitrary direction  $(l, m)$ ,  $\mathbf{B}$  is locally equivalent to the usual paraxial brightness matrix in Cartesian coordinates. From its definition it easy to see that  $\mathbf{B}$  is a Hermitian matrix.

By using the  $lmn$  and  $uvw$  spaces, as spanned by the vectors  $\mathbf{s}$ , and  $\mathbf{D}_\lambda$ , we can write equation (11) in a more explicit form. The exact form, though, depends on the extent of  $\mathcal{F}$ . If it is entirely in the hemisphere  $n > 0$ , then we write  $\mathcal{F} = \mathcal{F}_+$  and

$$\mathbf{V}^{(3)}(u, v, w) = \iint_{\mathcal{F}_+} \mathbf{T}\mathbf{B}\mathbf{T}^T \frac{e^{-i2\pi[ul+vm+w(n-1)]}}{n} dldm, \quad (23)$$

where we have used  $d\Omega = dldm/|n|$ , and where the matrices depend implicitly on  $l$  and  $m$ .

If, however, part of  $\mathcal{F}$  is in the  $n < 0$  hemisphere, then we must add to equation (23) the contribution from this hemisphere given by the integral

$$\iint_{\mathcal{F}_-} \mathbf{T}\mathbf{B}(l, m; n < 0) \mathbf{T}^T \frac{e^{-i2\pi[ul+vm+w(n-1)]}}{|n|} dldm, \quad (24)$$

where  $\mathcal{F}_-$  is the subset of  $\mathcal{F}$  in the  $n < 0$  hemisphere.

The image horizon,  $n = 0$ , can also be included by reparametrising the integral in terms of  $(m, n)$  rather than  $(l, m)$  and using the replacement  $d\Omega = dm dn / |l|$ . The poles  $m = \pm 1$  can also be imaged, but one must then stipulate the orientation of the  $\hat{\phi}$  and  $\hat{\theta}$  vectors at these singular points.

Now, by extending  $\mathcal{F}_+$  to cover the entire  $n > 0$  hemisphere and extending  $\mathcal{F}_-$  to cover the entire  $n < 0$  hemisphere, the entire celestial sphere can be imaged in a single telescope pointing. An assumption here is that  $\mathcal{M}$  is a proper three-dimensional volume, or in other words, the baselines should be non-coplanar. If  $\mathcal{M}$  is just a plane (coplanar baselines) then only one hemisphere can be mapped uniquely.

Equation (23) is our main result. It says that the full  $3 \times 3$  electric visibility matrix  $\mathbf{V}^{(3)}$  on the three-dimensional  $uvw$  space is given by the  $2 \times 2$  brightness matrix  $\mathbf{B}$  on the two-dimensional  $lm$  plane. The fact that this is a relationship between two matrices with different matrix dimensions is a fundamental feature of our vC-Z, and makes it clear that it is not just a matrix-valued generalisation of the wide-field, scalar vC-Z. Mathematically, this is ultimately due to the ranks of the fundamental matrices, of which we will speak more in section 4.

The remaining vC-Z relationships that form a complete characterisation of the electromagnetic coherence response of a radio astronomical interferometer are given in appendix A.

### 3 WIDE-FIELD vC-Z RELATION AS A TRANSFORM

In the previous section we derived a vC-Z relation, equation (23), in which the visibility matrix is determined from the brightness matrix. It is well known that the original vC-Z theorem (far-zone form) for narrow-fields states that there is a two-dimensional Fourier transform relationship between visibility and brightness. In astronomical interferometry the transform aspect of this relationship is exploited to produce brightness images from measured visibility. The wide-field vC-Z, equation (23), is not a two-dimensional Fourier transform, but, as we will now show, it is still possible invert it and thereby establish a sort of generalised transform.

First we should state that there are several ways of expressing  $\mathbf{B}$  in terms of  $\mathbf{V}^{(3)}$ , even though the wide-field vC-Z relation, equation (23), is a one-to-one relationship in general. This is because  $\mathbf{V}^{(3)}$  has redundancies, as one can expect considering the asymmetry in the respective matrix dimensions of  $\mathbf{B}$  and  $\mathbf{V}^{(3)}$ . So although the rank of  $\mathbf{V}^{(3)}$  is three in general, it is overdetermined if  $\mathbf{B}$  is known. We will first derive a solution that is valid for the entire celestial sphere based on the full  $\mathbf{V}^{(3)}$ . The case when only a projection of  $\mathbf{V}^{(3)}$  is available will be discussed in section 5.

Consider that we are given the full  $\mathbf{V}^{(3)}(u, v, w)$  and that we would like to solve equation (23) for  $\mathbf{B}$ . By extending the solution of the scalar problem described in Cornwell & Perley (1992) to the three-dimensional matrix relationship in equation (23) we find that one approximate solution is

$$\mathbf{B}^{(3)}(l, m; n > 0) = \sqrt{1 - l^2 - m^2} \times \int_0^1 \iiint_{\mathcal{M} - \mathcal{M}'} \mathbf{V}^{(3)}(u, v, w) e^{i2\pi[ul + vm + w(n-1)]} du dv dw dn. \quad (25)$$

where  $\mathcal{M} - \mathcal{M}'$  is the  $uvw$  space spanned by  $(\mathbf{r} - \mathbf{r}')/\lambda$  for  $\mathbf{r} \in \mathcal{M}$  and  $\mathbf{r}' \in \mathcal{M}$ . From  $\mathbf{B}^{(3)}$ , we can find a solution for the two-

dimensional brightness matrix,

$$\mathbf{B}(l, m; n > 0) = \mathbf{T}^T \mathbf{B}^{(3)} \mathbf{T}. \quad (26)$$

An analogous expression applies for  $\mathbf{B}^{(3)}(l, m; n < 0)$  but with the integration over  $n$  running from  $-1$  to  $0$  rather than  $0$  to  $1$ . For  $n = 0$ , an expression can be obtained by using the  $(m, n)$  parametrised vC-Z mentioned in the previous section.

We can now state the generalised transform as

$$\mathbf{V}^{(3)}(u, v, w) \rightleftharpoons \mathbf{B}(l, m) \quad (27)$$

where  $\rightleftharpoons$  reads “is wide-field, polarimetric vC-Z related to”. The relation from brightness matrix to visibility matrix is given (for  $n > 0$ ) by equation (23), and the relation from visibility matrix to brightness matrix is given by equations (26) and (25). Note that the  $\rightleftharpoons$  relation is not simply a two-dimensional Fourier transform as in the narrow-field case. Furthermore, the difference in the dimensionality of  $\mathbf{V}^{(3)}$  and  $\mathbf{B}$  make it clear that equation (27) cannot simply be a matrix generalisation of the scalar vC-Z relation, as is sometimes assumed.

### 4 USING STOKES PARAMETERS FOR WIDE FIELDS

In practice it is common to use Stokes parameters to characterise the brightness and visibility matrices over narrow fields. Let us see how Stokes parameters can be applied to the wide-field vC-Z relation, equation (23).

Let us first consider brightnesses. The difference between the two-dimensional brightness matrix  $\mathbf{B}$  in equation (23) and the usual two-dimensional brightness matrix in the paraxial approximation is that the former is defined on a spherical domain while the latter is defined on a Cartesian plane. Locally, for a source at some  $(l, m)$ , the two brightness matrices can be made equal. Thus we can define the Stokes parameters in terms of the components of  $\mathbf{B}$  with respect to a spherical basis  $\{\hat{\phi}, \hat{\theta}\}$  in analogy with Cartesian basis as

$$\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} B_{\theta\theta} + B_{\phi\phi} \\ B_{\theta\theta} - B_{\phi\phi} \\ B_{\theta\phi} + B_{\phi\theta} \\ +i(B_{\theta\phi} - B_{\phi\theta}) \end{pmatrix}. \quad (28)$$

When the spherical system is aligned as it is in section 2 such that the intersection of its central meridian and its equator is located at the centre of the FoV and the pole  $A$  is towards the Earth’s North pole, then in a sufficiently narrow field around the centre of the FoV the Stokes brightnesses  $(I, Q, U, V)^T$  are approximately equal to the Stokes parameters of the IAU (Hamaker & Bregman 1996). If one wishes to conform with IAU Stokes parameters over the entire celestial sphere, then one can use equation (28) with  $\mathbf{T}$  replaced by  $\mathbf{T}'(\Theta)$ , equation (B6), where  $\Theta$  is the declination of the centre of the FoV. The correspondence is then that  $+\hat{\theta}$  is the IAU’s  $+\hat{x}$ , and  $+\hat{\phi}$  is the IAU’s  $+\hat{y}$ .

Note however that the IAU basis set for polarimetry is not the same as the Cartesian basis set widely used in radio interferometry for defining source directions and baselines, which in this paper is denoted  $\{\hat{x}, \hat{y}, \hat{z}\}$ . In adopting both these systems, therefore, a sacrifice must be made, and we have chosen to slightly modify the usual Pauli matrices that are used to relate the brightness matrix to the Stokes parameters. Explicitly, the brightness matrix in terms of the Stokes parameters in equation (28) is in this paper

$$\mathbf{B} = \frac{1}{2} \begin{pmatrix} I - Q & U + iV \\ U - iV & I + Q \end{pmatrix}. \quad (29)$$

As can be seen, the expansion of this expression into unitary matrices, as in Hamaker (2000, eq. 6), leads to matrices equivalent to Pauli matrices but with a change of overall sign for the matrices associated with  $Q$  and  $V$ .

In terms of equation (28), we can write the three-dimensional brightness matrix in equation (23) as

$$\mathbf{B}^{(3)} = \mathbf{T}\mathbf{B}\mathbf{T}^T = \mathbf{I}\mathbf{C}_I + \mathbf{Q}\mathbf{C}_Q + \mathbf{U}\mathbf{C}_U + \mathbf{V}\mathbf{C}_V, \quad (30)$$

where

$$\begin{aligned} \mathbf{C}_I &= \frac{1}{2} \begin{pmatrix} 1-l^2 & -lm & -ln \\ -lm & 1-m^2 & -mn \\ -ln & -mn & l^2+m^2 \end{pmatrix} \\ \mathbf{C}_Q &= \frac{1}{2} \begin{pmatrix} \frac{-n^2+l^2m^2}{1-m^2} & -lm & \frac{(1+m^2)ln}{1-m^2} \\ -lm & 1-m^2 & -mn \\ \frac{(1+m^2)ln}{1-m^2} & -mn & \frac{-l^2+m^2n^2}{1-m^2} \end{pmatrix} \\ \mathbf{C}_U &= \frac{1}{2} \begin{pmatrix} -2\frac{lmn}{1-m^2} & n & \frac{m(l^2-n^2)}{1-m^2} \\ n & 0 & -l \\ \frac{m(l^2-n^2)}{1-m^2} & -l & 2\frac{lmn}{1-m^2} \end{pmatrix} \\ \mathbf{C}_V &= i\frac{1}{2} \begin{pmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{pmatrix}. \end{aligned}$$

It is easy to see that the  $\mathbf{C}_i$  for  $i = I, Q, U, V$  depend only on  $(l, m, n)$ , and that they become the three-dimensional analogs of the equivalent Pauli matrices in equation (29).

Equation (30) shows that for every look-direction  $(l, m)$ , the three-dimensional brightness matrix has four degrees of freedom, here expressed as the four Stokes parameters. In other words, the four Stokes parameters completely characterise the partially polarized brightness also for wide fields, under the assumptions made in the derivation of equation (23)).

Now let consider if Stokes visibilities can be extended to wide fields. A consequence of the three-dimensionality of  $\mathbf{V}^{(3)}$  in equation (23) is, however, that the Stokes parameters cannot in general fully characterise the electric visibility. This is because, in contrast to  $\mathbf{B}^{(3)}$ , which has a rank of at most two due to the transversality condition<sup>3</sup>, equation (12),  $\mathbf{V}^{(3)}$  has no similar constraint. Indeed one can convince oneself of the full rank of  $\mathbf{V}^{(3)}$  by considering two distinct point sources: the weighted sum of their  $\mathbf{B}^{(3)}$  matrices at some  $(u, v, w)$  point according to equation (23), will in general be rank three. As there are only four Stokes parameters, albeit complex-valued in the case of visibilities, they cannot fully parametrise a rank three matrix.

Alternatively, rather than use the standard four Stokes parameters, one could also use complexified versions of the nine, real, generalised Stokes parameters (Carozzi et al. 2000). These parameters are analogous to the standard Stokes parameters but can completely describe the coherence of the full three-dimensional electric field. In light of the discussion above, these generalised Stokes parameters are particularly suitable for parametrisation of  $\mathbf{V}^{(3)}$ , but we will not discuss them here any further.

<sup>3</sup> That equation (12) implies that  $\mathbf{B}^{(3)}$  has rank two comes from the rank-nullity theorem of linear algebra, since equation (12) implies that dimension of the null space is one and the matrix dimension of  $\mathbf{B}^{(3)}$  is three, so  $3-1=2$  is the rank of  $\mathbf{B}^{(3)}$ .

## 5 GENERALISED MEASUREMENT EQUATIONS

The vC-Z theorem is a basic, fundamental physical relationship is independent of technology. The measurement equation (M.E.) of radio astronomy, on the other hand, includes practical aspects of telescope measurements, in particular the instrumental response of the telescope. Usually it is a relationship between a  $2 \times 2$  brightness matrix and a  $2 \times 2$  cross-correlation matrix of the output-voltage of a dual-polarized interferometer. Since in the past such two-dimensional M.E. have been tacitly based on the paraxial approximation valid only for narrow fields, it is important to verify that it can be recovered from the wide-field, polarimetric vC-Z, equation (23), for which the paraxial approximation is not used. Although it is possible to do this in a simple, straightforward way, we choose to do it in a more detailed way, introducing a formalism that extends the usual two-dimensional, electric field based model of radio astronomical antenna response.

### 5.1 Electromagnetic antenna response model

To obtain a 2-D M.E. we must first introduce a formalism for converting the full electric field to a voltage in the interferometer antenna. An electric field  $\mathbf{E}$  at an antenna excites an open circuit voltage  $V$ . Assuming linearity, these two quantities are related as

$$V = \mathbf{L} \cdot \mathbf{E} \quad (31)$$

where  $\mathbf{L}$  is the antenna effective length vector. In general it is a function of incidence direction, i.e.  $(l, m)$ , but here we will only use ideal, Hertzian dipole antennas (short electric dipoles). These have the important property that their effective length does not depend on incidence direction,  $\mathbf{L}$  is just a constant unit vector, and so it directly samples the component of the electric field along its length.

If we have  $n$  co-located antennas that have no mutual coupling, their output voltages can be written in a matrix form

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} = \begin{pmatrix} L_x^{(1)} & L_y^{(1)} & L_z^{(1)} \\ L_x^{(2)} & L_y^{(2)} & L_z^{(2)} \\ \vdots & \vdots & \vdots \\ L_x^{(n)} & L_y^{(n)} & L_z^{(n)} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \quad (32)$$

where the  $i$ -th row in the  $n \times 3$  matrix contains the components of antenna effective length vector  $\mathbf{L}^{(i)}$ . The matrix of antenna effective lengths, denoted  $\mathbf{L}$ , has physical dimension length and is a  $n$ -dimensional extension of the  $\mathbf{Q}$  matrix in Hamaker et al. (1996, eq. (2)), for which  $n = 2$  and thus models dual-polarized antennas. When  $n = 3$  and the antennas are linearly independent, then they sample the full three-dimensional electric field. Such an antenna system is called *tri-polarized* antenna in general, and tripole antenna if the three dipoles are approximately mutually orthogonal. One can also include antennas that sample the magnetic field, and an arrangement of electric and magnetic antennas can be constructed so as to sample the full electromagnetic field at a point. Such antennas are called electromagnetic vector-sensors (Nehorai & Paldi 1991; Bergman et al. 2005). In what follows, we will only be interested in dual- or tri-polarized antenna systems.

In particular, let us consider a dual-polarized antenna that consists of two co-located, non-mutually coupled dipole antennas, one aligned along  $\hat{\mathbf{x}}$  and the other along  $\hat{\mathbf{y}}$ . The response of such a dual-polarized antenna is

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = L \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = L \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (33)$$

where  $L$  is the antenna effective length. We have changed the subscripts on the voltages to reflect the right-hand side of the equation, in other words, for this dual-polarized antenna, each voltage component is directly proportional to a unique Cartesian component of the electric field regardless of the radiations incidence angle. Because of this property, it can be regarded as an ideal dual-polarized polarimeter element. Associated with each dual-polarized antenna element is a two-dimensional plane in which the polarization is defined and measured. If this plane is the same for all of the elements in a dual-polarized interferometer or if all the planes are mutually parallel (common design goal for polarimetric interferometers) we will say that such an interferometer is *plane-polarized*, in analogy with plane-polarized waves. Note that a plane-polarized interferometer is not necessarily a co-planar interferometer, and that a non-plane-polarized interferometer may be co-planar. If the plane of a plane-polarized interferometer is to be specified explicitly, we will say, e.g. in the case of equation (33), that it is *xy*-polarized.

Although most existing polarimetric interferometers in radio astronomy are intended to be plane-polarized, it is possible to have more general antenna elements such as electromagnetic vector-sensor arrays or electric tripole arrays. An real-life example of the latter is the LOIS test station, see Bergman, Carozzi & Karlsson (2003); Bergman et al. (2005); Bergman & Carozzi (2008); Thidé (2004); Guthmann & Thidé (2005). The prime motivation for such a tri-polarized system is that it samples the full electric field in a single telescope pointing rather than just a projection.

## 5.2 Recovering the 2-D M.E.: dual-polarized antennas and the paraxial limit

Now that we have introduced a model formalism for antenna response we can derive the 2-D M.E. for the special but important case of the *xy*-polarized (Hertzian dipole) interferometer, that is, the polarization plane is normal to the centre of the FoV. The output voltages from the *xy*-polarized elements located at points  $P_1$  and  $P_2$  are cross-correlated and the result can be expressed as the correlation matrix  $R_{ij}(1, 2) = \langle V_i(1)V_j^*(2) \rangle$  for  $i, j = x, y$ , where the arguments 1 and 2 refer to baseline points  $P_1$  and  $P_2$ .  $\mathbf{L}$  is matrix multiplied from the left and its transpose from the right with  $\mathbf{B}^{(3)}$ , but since it does not depend on  $(l, m)$  in this case, it can be pulled out of the integral in (23), and so the correlator output can be written

$$\mathbf{R} = \mathbf{L}^{(xy)} \mathbf{V}^{(3)} \left( \mathbf{L}^{(xy)} \right)^T,$$

where

$$\mathbf{L}^{(xy)} = L \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (34)$$

is the *xy*-polarized antenna elements, effective length matrix. As one can see, the effect of  $\mathbf{L}^{(xy)}$  is equivalent to projecting the field vectors into the *xy*-plane and multiplying by  $L$ . In terms of the brightness matrix the correlator output is

$$\mathbf{R} = |L|^2 \times \iint_{\mathcal{F}} \mathbf{T}^{(xy)} \mathbf{B} \left( \mathbf{T}^{(xy)} \right)^T \frac{e^{-i2\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]}}{\sqrt{1-l^2-m^2}} dldm \quad (35)$$

where we have introduced the *xy*-projected transformation matrix

$$\mathbf{T}^{(xy)} = \frac{1}{L} \mathbf{L}^{(xy)} \mathbf{T} = \frac{1}{\sqrt{1-m^2}} \begin{pmatrix} n & -lm \\ 0 & 1-m^2 \end{pmatrix}$$

to simplify the final result, which is now clearly two-dimensional. Equation (35) is a wide-field M.E. but with the novel Jones matrix  $\mathbf{T}^{(xy)}$  that physically represents a projection of the three-dimensional electric field vector onto the *xy*-plane.

In the narrow FoV limit  $\sqrt{l^2+m^2} \ll n$ , so we can approximate  $\mathbf{T}^{(xy)}$  in equation (35) up to first order in  $l$  and  $m$  by the two-dimensional unity matrix, and so

$$\mathbf{R} = |L|^2 \iint_{\mathcal{F}} \mathbf{B} e^{-i2\pi(ul+vm)} dldm \quad \text{for } \sqrt{l^2+m^2} \ll n. \quad (36)$$

This is the basic M.E. of astronomical interferometry, see Thompson et al. (2001, eq. (14.7)). Thus we have shown that the wide-field, polarimetric vC-Z, equation (23) indeed reduces to the usual two-dimensional, Jones vector based M.E. in the paraxial limit. The result, equation (36), depended on the particular the spherical coordinate system used as default in this paper. Only this particular choice reduces directly to the standard 2-D M.E. in the paraxial limit.

To see what the dual-polarized M.E. equation (36) misses by not including the third dimension along  $z$ , let us go back to equation (23) and let assume the paraxial approximation,  $\sqrt{l^2+m^2} \ll n$ . In this case

$$\mathbf{T} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -l & -m \end{pmatrix}, \quad (37)$$

where we have kept only terms of first order in  $l$  and  $m$ .  $\mathcal{V}_{ij}^{(3)}$  in equation (23) is identical to  $R_{ij}/|L|^2$  in equation (36) for  $i, j = x, y$ , but for the  $z$  components,

$$\mathcal{V}_{iz}^{(3)} \approx - \iint_{\mathcal{F}} \mathbf{B} \cdot \begin{pmatrix} l \\ m \end{pmatrix} e^{-i2\pi(ul+vm)} dldm \quad \text{for } i = x, y \quad (38)$$

and

$$\mathcal{V}_{zz}^{(3)} \approx 0. \quad (39)$$

So the  $zz$  component of the three-dimensional visibility matrix does not provide anything, but the  $xz$  (and  $zx$ ) and the  $yz$  (and  $zy$ ) components do. Thus even for a narrow FoV, a dual-polarized interferometer does not measure the full set of generally non-zero electric visibilities. Note that if the antenna array is not exactly plane-polarized, these additional visibility components will contribute to the output-voltage of such an array.

This leads to the important question of how serious the loss of visibility information is in a dual-polarized interferometer. More specifically, we ask whether a plane-polarized interferometer, given by some  $2 \times 3$  matrix  $\mathbf{L}$ , can in general perform full polarimetry of an arbitrary source distribution. The answer is that the source brightness matrix in some direction can be determined fully only if

$$\det(\mathbf{L}\mathbf{T}) \neq 0$$

since in this case  $\mathbf{L}\mathbf{T}$  is invertible. For the special but important case  $\mathbf{L} = \mathbf{L}^{(xy)}$  this condition is equivalent to

$$l^2 + m^2 \neq 1.$$

Thus, an *xy*-polarized interferometer can recover the full polarimetry except on the great circle orthogonal to  $C$ , which we may call the imaging horizon. So, in wide-field imaging with *xy*-polarized interferometers the image horizon cannot be measured with a single telescope pointing. Under noise-free conditions, this would pose

little problem, but if we add in the effects of noise then the great circle of directions for which full polarimetry is not feasible broadens as a function of the signal-to-noise ratio.

Although the discussion above was mainly focused on plane-polarized interferometers, the three-dimensional formalism developed here can also be applied to more general dual-polarized interferometers. In particular it can model the situation when the polarization planes of dual-polarized antennas in an array are not all parallel. Since several large arrays of dual-polarized antenna based interferometers are currently being planned for, this would address the important question of whether such arrays should strive to be plane-polarized or whether they should purposely not be plane-polarized to minimize the inversion problems mentioned above.

### 5.3 Polarization aberration in $xy$ -polarized, Hertzian dipole interferometers

We now show that the M.E. for the  $xy$ -polarized, Hertzian dipole interferometer, equation (35), exhibits distortions that depend on the look-direction, that is, the images based on these brightnesses contain polarization aberrations (McGuire & Chipman 1990). This agrees with the general understanding in observational radio astronomy that the polarimetry of a telescope is worse off-axis than on-axis.

Say we have measured the  $2 \times 2$  correlation matrix  $\mathbf{R}(u, v, w)$ . We cannot use the formal solution (25) directly since it is for the  $3 \times 3$  visibility, and it is not clear how to obtain the remaining, unmeasured  $z$  components from the  $x$  and  $y$  components of  $\mathbf{R}$ . On the other hand, for the scalar case, the formal solution for producing a synthesized image is the following scalar, wide-field imaging equation:

$$I = \frac{\sqrt{1-l^2-m^2}}{|L|^2} \iiint \mathcal{V} e^{2\pi i[ul+vm+w(n-1)]} du dv dw dn, \quad (40)$$

where  $I = I(l, m)$  is the scalar brightness and  $\mathcal{V} = \mathcal{V}(u, v, w)$  is the scalar visibility, see equation (13) in Cornwell & Perley (1992) who call it the 3D method of synthesis imaging. Let us apply (40) to each component of  $\mathbf{R}$  as if it were a scalar, thus creating a matrix analogue of the scalar, wide-field imaging equation. The resulting brightness matrix  $\mathbf{B}^{(xy)}$  (polarized image) is

$$\mathbf{B}^{(xy)} = \frac{\sqrt{1-l^2-m^2}}{|L|^2} \iiint \mathbf{R} e^{2\pi i[ul+vm+w(n-1)]} du dv dw dn. \quad (41)$$

However, this is not the true brightness matrix because we see from equation (35) that

$$\mathbf{B}^{(xy)} = \mathbf{T}^{(xy)} \mathbf{B} (\mathbf{T}^{(xy)})^T = \frac{1}{|L|^2} \mathbf{L}^{(xy)} \mathbf{B}^{(3)} (\mathbf{L}^{(xy)})^T. \quad (42)$$

It follows that  $\mathbf{B}^{(xy)}$  is actually the projection of the three-dimensional brightness matrix into the  $xy$ -plane. The Stokes  $xy$ -projected brightnesses  $\mathbf{S}_P$  are based directly on  $\mathbf{B}^{(xy)}$  in analogy with equation (28), that is,

$$\mathbf{S}_P(l, m) = \begin{pmatrix} I_P \\ Q_P \\ U_P \\ V_P \end{pmatrix} = \begin{pmatrix} B_{xx}^{(xy)} + B_{yy}^{(xy)} \\ B_{xx}^{(xy)} - B_{yy}^{(xy)} \\ B_{xy}^{(xy)} + B_{yx}^{(xy)} \\ i(B_{xy}^{(xy)} - B_{yx}^{(xy)}) \end{pmatrix}. \quad (43)$$

The relationship between these Stokes brightnesses and the true Stokes brightnesses  $\mathbf{S}$ , which are based on  $\mathbf{B}$  can be found by recasting

equation (42) as

$$\mathbf{S}_P = \mathbf{M} \mathbf{S}, \quad (44)$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2}(1+n^2) & -\frac{m^2}{2} + \frac{(1+m^2)l^2}{2(1-m^2)} & -\frac{lmn}{1-m^2} & 0 \\ \frac{1}{2}(l^2-m^2) & 1 - \frac{m^2}{2} - \frac{(1+m^2)l^2}{2(1-m^2)} & \frac{lmn}{1-m^2} & 0 \\ -lm & -lm & n & 0 \\ 0 & 0 & 0 & n \end{pmatrix}. \quad (45)$$

$\mathbf{M}$  is a Mueller matrix that quantifies the distortion of true Stokes brightnesses based on the imaging equation (41).

For a very narrow FoV,  $(l, m) \approx (0, 0)$  and  $\mathbf{M}$  is approximately unity. In general, however,  $\mathbf{M}$  is not the unit matrix with the effect that the perceived Stokes vector is a distortion of the true Stokes vector. Thus, without further processing a plane-polarized interferometer of short dipoles will exhibit polarization aberrations over wide fields. By contrast, a tripole array interferometer is, at least in theory, polarimetrically aberration-free over a wide field, since the scalar wide-field imaging equation (40) applied to the components of its visibility matrix as if they were scalars gives  $\mathbf{B}^{(3)}$ , which can be interpreted as the exact  $\mathbf{B}$  in Cartesian (rather than spherical) components.

Examples of these wide-field distortions are displayed in Fig. 2. It shows aberration effects for source distributions that are constant over the entire hemisphere, by which we mean that  $\mathbf{B}(l, m) = \mathbf{B}$ , so the brightnesses do not explicitly vary with direction<sup>4</sup>. Fortunately, these distortions can be compensated for in the image plane since  $\mathbf{M}$  is invertible and well-conditioned as long as  $l^2 + m^2$  is not close to one.

Another aspect of equation (44) that is important, is that it shows, not surprisingly, that partially polarized, wide fields cannot be treated as scalar, wide fields. In fact, even for the supposedly ‘scalar’ case, that is, when we only consider the scalar visibility  $\mathcal{V}(u, v, w) = \frac{1}{2|L|^2} \text{Tr}(\mathbf{R})$  of an unpolarized source distribution, so  $Q = U = V = 0$ , equations (44) and (41) imply that

$$\mathcal{V}(u, v, w) \Rightarrow I(l, m) \frac{2 - l^2 - m^2}{2\sqrt{1-l^2-m^2}}, \quad (46)$$

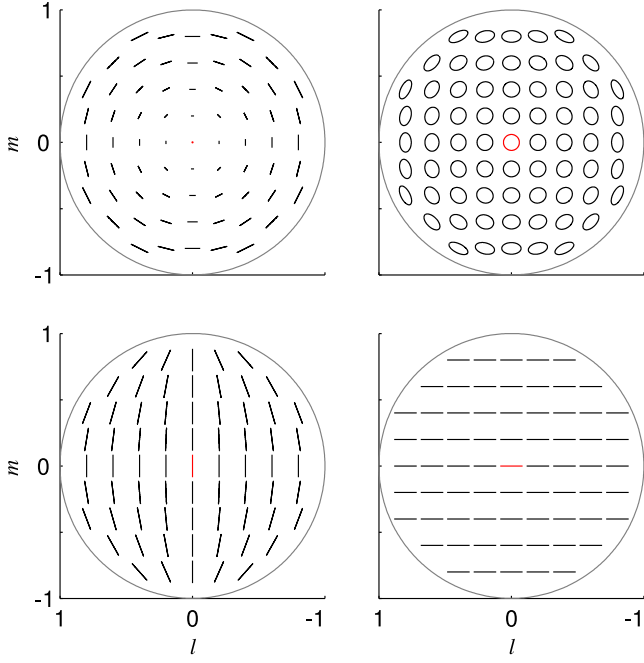
where  $\Rightarrow$  stands for two-dimensional Fourier transform. This should be compared with  $\mathcal{V}(u, v, w) \Rightarrow I(l, m)/\sqrt{1-l^2-m^2}$  for the isotropic, scalar antenna case (see, e.g., equation (1) in Cornwell & Perley (1992)). This expression differs even for narrow fields since, to lowest, non-vanishing order in  $l$  and  $m$ , equation (46) becomes approximately

$$\mathcal{V}(u, v, w) \Rightarrow I(l, m) \left[ 1 - \frac{1}{4}(l^2 + m^2)^2 \right], \quad (47)$$

while  $\mathcal{V} \Rightarrow I(1 - 1/2(l^2 + m^2))$  for the isotropic, scalar antenna case. Thus short dipoles are less aberrated than isotropic, scalar antennas.

The conclusion here, that scalar theory is not sufficient for the description of the general vC-Z relations, agrees with those detailed in the field of optical coherence, see e.g. Carter (1980); Saastamoinen et al. (2003). Also, equation (46) corresponds to an analogous equation in Saastamoinen et al. (2003). Consider an unpolarized point source at  $(l, m, n) = (l', m', n')$ , so  $I(l, m) =$

<sup>4</sup> This does not mean that the distributions are necessarily isotropic since they can still vary through their reference to the (direction-dependent) basis  $\{\hat{\phi}, \hat{\theta}\}$ .



**Figure 2.** Distortion of various polarized source distributions across the hemisphere for an  $xy$ -polarized interferometer. All distributions are such that the Stokes brightnesses are constant, that is, they do not vary explicitly with direction although they may vary implicitly due to variation of the reference system for the Stokes brightnesses,  $\hat{\theta}$  and  $\hat{\phi}$ , with direction. The plots are of the polarization ellipses corresponding to the normalized Stokes parameters  $(Q_p/I_p, U_p/I_p, V_p/I_p)$ , where  $(I_p, Q_p, U_p, V_p)$  are the  $xy$ -projected Stokes brightnesses as a function of the direction  $(l, m)$ . The plots clearly show polarization aberration, that is, a direction dependent distortion in the observed polarization. The source distributions state of polarization (shown in red) can be seen at the centre of the FoV,  $(l, m) = (0, 0)$ , where there is no distortion. The upper-left panel shows  $\mathbf{S} = (1, 0, 0, 0)$ , i.e. completely unpolarized radiation, the upper-right panel shows circularly polarized radiation, the lower-left shows radiation linearly polarized along  $\hat{\theta}$ ,  $\mathbf{S} = (1, 1, 0, 0)$ , and the lower-right panel is for radiation linearly polarized along  $\hat{\phi}$ ,  $\mathbf{S} = (1, -1, 0, 0)$ .

$I\delta(l-l', m-m')|n|$ , and express  $n'$  as  $\cos \alpha$  (where  $\alpha$  is the angle between the point source position and  $C$ , and is equivalent to  $\theta$  in Saastamoinen et al. (2003)), then the intensity of the unpolarized point source, as measured by single-pixel telescope, is  $\mathcal{V}(0, 0, 0) = \frac{1}{2}I(1 + \cos^2 \alpha)$ . This corresponds to equation (47) in Saastamoinen et al. (2003).

Although the results will be different for other types of antennas, the Hertzian dipole is an important special case as it is the simplest polarimetric antenna and they are directly proportional to the Cartesian coordinates of the electric visibility matrix.

The ultimate reason for the aberration is of course that the incident, transverse field is being projected onto the polarization plane of the Hertzian dipoles thereby distorting the field. Indeed this projection is identical to the orthographic projection of a hemisphere in cartography. In comparison to other effects encountered in instrumental calibration, such as beam-shape, these effects may seem small. They are, however, important in that they determine the ultimate limits of polarimetry since these effects are of an intrinsic, geometric nature.

Inspection of  $\mathbf{M}$  reveals that around  $(l, m) = (0, 0)$  the aberrations are  $O(l^2, m^2)$ . In the case of the aperture array of the SKA, the preliminary specification (Schilizzi et al. 2007) has a mean FoV

of 250 square degrees, so the aberration error is in the order of  $\sim 2\%$ , or  $-16$  dB, at the edge of the FoV. This should be compared with the requirements from the key scientific projects that in some cases call for  $-30$  dB in polarization purity over wide fields. Thus, these polarization aberrations will need to be considered in wide-field imaging with SKA.

It is not difficult to show that these polarization aberrations occurring at the edges of wide-field images also occur for off-axis imaging with fixed mount telescopes. This occurs, for instance, in phased arrays of crossed dipoles such as LOFAR and the low frequency part of SKA, where imaging at scan angles away from bore-sight (zenith) will be achieved by electronically steering the beam-form. Since the crossed dipoles are fixed to the ground and are not on a mechanically rotating mount, a situation geometrically analogous to the wide-field imaging consider above occurs. This leads to similar aberrations in polarization for short electric dipole arrays. Scan angles of  $45^\circ$  have been proposed for which the aberrations will of the order of  $-3$  dB.

## 6 CONCLUSION

We have derived the full set of electromagnetic vC-Z relations, which are the basis of radio astronomical interferometry, without invoking the paraxial approximation. These relations allow all-sky imaging in a single telescope pointing. We have achieved these relations by generalising the usual 2-D Cartesian Jones vector based M.E. to a 3-D Wolf coherence matrix formulation on the (celestial) sphere. The derived wide-field vC-Z relations are not simply trivial matrix (or vector) analogues of the wide-field, scalar vC-Z, they exhibit direction dependent polarimetric effects. Indeed even in the scalar limit (that is, unpolarized radiation), our M.E. is not the same as the M.E. derived from scalar theory, in the case of Hertzian dipoles. Furthermore, we found that, for an arbitrary wide-field of sources, the electric visibilities generally have nine complex components for an arbitrary baseline. This implies that the standard Stokes (Cartesian) visibilities do not provide a full description of electric coherence for wide fields. We have also shown that our vC-Z relation (for the electric field) reduces to the standard 2-D M.E. after a 2-D projection. We showed that a consequence of this projection is that plane-polarized, Hertzian dipole interferometers are aberrated polarimetrically. Fortunately, these aberrations can be corrected for in the image plane for sources sufficiently far away from the plane of the dual-polarized antenna elements.

Besides its use in the derivation of the wide-field vC-Z relation, we believe that the 3-D Wolf formalism can be useful in constructing more general 3-D M.E. in cases requiring the full set of electric components. As examples, we mention the modelling of tri-polarized element arrays, the modelling of dual-polarized element arrays that are not strictly plane-polarized (due to manufacturing errors or the Earth's curvature), and the modelling of propagation that is not along the line-of-sight (due to refraction or diffraction in the ionosphere, e.g.).

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## APPENDIX A: FULL ELECTROMAGNETIC vC-Z RELATIONS

In the previous sections we considered only the electric field and its auto-correlation, but now we look at the full electromagnetic field. One motivation to do this is that, especially for low radio frequencies, it is possible to sample both the electric and the magnetic fields and thereby measure electromagnetic coherence fully, see Bergman, Carozzi & Karlsson (2003), Bergman et al. (2005) or Bergman & Carozzi (2008). Such sensors have been deployed in some radio interferometers and will possibly provide unique and novel astronomical measurements. For completeness then, we present the rest of the electromagnetic correlations analogous to the electric vC-Z relation in equation (27).

We found previously that the  $3 \times 3$  electric visibility matrix was related to the  $2 \times 2$  electric brightness matrix as

$$\mathbf{V}^{(EE)}(u, v, w) \leftrightarrow \mathbf{B}^{(ee)}(l, m) \quad (\text{A1})$$

where  $\leftrightarrow$  denotes the vC-Z relationship given by equation (23). Note that we have now changed the notation of  $\mathbf{V}^{(3)}$  to  $\mathbf{V}^{(EE)}$  and  $\mathbf{B}$  to  $\mathbf{B}^{(ee)}$  to indicate that these quantities represent auto-correlations of visibility electric field and the brightness Jones vectors, respectively. A full electromagnetic vC-Z relationship between brightnesses and visibilities requires also the auto-correlations of the magnetic field and the cross-correlation between the electric and magnetic fields, see Mandel & Wolf (1995, chap. 6). We define the electromagnetic visibilities for baseline  $\mathbf{D}_\lambda$  with respect to the phase reference direction  $\mathbf{s}_0$  as

$$\mathcal{V}_{ij}^{(EE)}(\mathbf{D}_\lambda) = \langle E_i(\mathbf{r}) E_j^*(\mathbf{r} - \mathbf{D}) \rangle \exp(i2\pi \mathbf{s}_0 \cdot \mathbf{D}_\lambda) \quad (\text{A2})$$

$$\mathcal{V}_{ij}^{(EH)}(\mathbf{D}_\lambda) = \langle E_i(\mathbf{r}) H_j^*(\mathbf{r} - \mathbf{D}) \rangle \exp(i2\pi \mathbf{s}_0 \cdot \mathbf{D}_\lambda) \quad (\text{A3})$$

$$\mathcal{V}_{ij}^{(HE)}(\mathbf{D}_\lambda) = \langle H_i(\mathbf{r}) E_j^*(\mathbf{r} - \mathbf{D}) \rangle \exp(i2\pi \mathbf{s}_0 \cdot \mathbf{D}_\lambda) \quad (\text{A4})$$

$$\mathcal{V}_{ij}^{(HH)}(\mathbf{D}_\lambda) = \langle H_i(\mathbf{r}) H_j^*(\mathbf{r} - \mathbf{D}) \rangle \exp(i2\pi \mathbf{s}_0 \cdot \mathbf{D}_\lambda) \quad (\text{A5})$$

for  $i, j = x, y, z$ , where  $\mathbf{H}(\mathbf{r})$  is the magnetic field at  $\mathbf{r}$ . The electromagnetic brightnesses in direction  $\mathbf{s}$  are defined as

$$B_{ij}^{(ee)}(\mathbf{s}) = \langle e_i(\mathbf{s}) e_j^*(\mathbf{s}) \rangle \quad (\text{A6})$$

$$B_{ij}^{(eh)}(\mathbf{s}) = \langle e_i(\mathbf{s}) h_j^*(\mathbf{s}) \rangle \quad (\text{A7})$$

$$B_{ij}^{(he)}(\mathbf{s}) = \langle h_i(\mathbf{s}) e_j^*(\mathbf{s}) \rangle \quad (\text{A8})$$

$$B_{ij}^{(hh)}(\mathbf{s}) = \langle h_i(\mathbf{s}) h_j^*(\mathbf{s}) \rangle, \quad (\text{A9})$$

for  $i, j = \phi, \theta$ , where  $\mathbf{h}(\mathbf{s})$  is the magnetic field from the source distribution in direction  $\mathbf{s}$ .

The magnetic field associated with the visibilities can be derived by applying Faraday's law,

$$\mathbf{H}(\mathbf{r}, t) = -i \frac{c}{2\pi\nu Z_0} \nabla \times \mathbf{E}(\mathbf{r}, t),$$

where  $Z_0$  is the impedance of free space, to the electric field used in derivation in section 2. The result is that the magnetic analogues of the electric vC-Z expressions involve a magnetic Jones vector  $\mathbf{h} = (h_\phi, h_\theta)^T$  that is related to the electric Jones vector through

$$-\frac{1}{Z_0} \mathbf{s} \times (\mathbf{T}\mathbf{e}) = \mathbf{T}\mathbf{F}\mathbf{e} = \mathbf{T}\mathbf{h}$$

where

$$\mathbf{F} = \frac{1}{Z_0} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A10})$$

or in other words

$$\mathbf{h} = \frac{1}{Z_0} \begin{pmatrix} +e_\theta \\ -e_\phi \end{pmatrix}.$$

This says that the magnetic Jones vector is directly determined from the electric Jones vector, and so no other independent electromagnetic source coherence statistics exist (in the far-field zone) other than the  $2 \times 2$  electric brightness matrix.

Repeating the derivation in section 2 but for the magnetic field we find we can write the rest of the vC-Z relations as

$$\mathcal{V}^{(EH)} \rightleftharpoons \mathbf{B}^{(eh)} = \mathbf{B}^{(ee)} \mathbf{F}^T \quad (\text{A11})$$

$$\mathcal{V}^{(HE)} \rightleftharpoons \mathbf{B}^{(he)} = \mathbf{F} \mathbf{B}^{(ee)} \quad (\text{A12})$$

$$\mathcal{V}^{(HH)} \rightleftharpoons \mathbf{B}^{(hh)} = \mathbf{F} \mathbf{B}^{(ee)} \mathbf{F}^T. \quad (\text{A13})$$

Thus one can see  $\mathbf{F}$  as a Jones-like matrix that switches between electric and magnetic coherencies.

These relations provide a complete description of the second-order coherence of the electromagnetic radiation. So, for instance, one can easily compute the Poynting visibility vector using the above relations,

$$\begin{aligned} \langle \Re [\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r} - \mathbf{D})] \rangle = \\ - \frac{1}{Z_0} \iint I \text{se}^{-i2\pi[ul+vm+w(n-1)]} d\Omega \end{aligned} \quad (\text{A14})$$

where  $I = I(l, m) = |e_\theta|^2 + |e_\phi|^2$ . This says that the power flux visibility vector is vC-Z related to the Stokes brightness propagating from sources.

## APPENDIX B: COORDINATE TRANSFORMATIONS

In this paper we have used the standard definitions of the  $uvw$ - and  $lmn$ -spaces, see Thompson et al. (2001), and we have used a spherical basis  $\{\hat{\theta}, \hat{\phi}\}$ . Both these coordinate systems are relative to the centre of the FoV. Often one wants to transform to some other system, and some details can be found in the standard texts (Thompson et al. 2001). However, what is not usually done explicitly is the transformation of the non-scalar brightnesses and visibilities. We will now show how to rotate the matrix brightnesses and visibilities found in the polarimetric, wide-field vC-Z, equation (23).

Equation (23) uses the Cartesian  $\{\hat{x}, \hat{y}, \hat{z}\}$  system as a basis for  $\mathcal{V}^{(3)}$ ,  $\mathbf{s}$ ,  $\mathbf{D}_\lambda$  and the rows of  $\mathbf{T}$ . It uses the spherical base vectors  $\{\hat{\theta}, \hat{\phi}\}$  as a basis for  $\mathbf{B}$  and the columns of  $\mathbf{T}$ . Under a rotation given by the  $3 \times 3$  orthogonal matrix  $\mathbf{Q}$ , the vectors and matrices in the Cartesian system transform in the usual manner:

$$\mathbf{s}' = \mathbf{Q}\mathbf{s}, \quad (\text{B1})$$

$$\mathbf{D}'_\lambda = \mathbf{Q}\mathbf{D}_\lambda, \quad (\text{B2})$$

$$\mathcal{V}^{(3)}(\mathbf{s}') = \mathbf{Q}\mathcal{V}^{(3)}(\mathbf{s})\mathbf{Q}^T. \quad (\text{B3})$$

Note that a transformation analogous to equation (B3) for the  $2 \times 2$  matrix  $\mathbf{R}$ , discussed in section 5.2, does not exist since  $\mathbf{R}$  lacks one of the dimensions necessary for a general coordinate transformation. Only the correlator output from a tri-polarized antenna array can be fully transformed in general.

The brightness matrix rotates as

$$\mathbf{B}'(\mathbf{s}') = \mathbf{T}^T(\mathbf{s}')\mathbf{Q}\mathbf{T}(\mathbf{s})\mathbf{B}(\mathbf{s})\mathbf{T}^T(\mathbf{s})\mathbf{Q}^T\mathbf{T}(\mathbf{s}'), \quad (\text{B4})$$

where  $\mathbf{s}$  on the left-hand side should be converted into  $\mathbf{s}'$  using equation (B1).

Actually,  $\mathbf{Q}$  need not be the same in equations (B3) and (B4), since the spherical and Cartesian systems can be rotated separately. This can be used to change the relative alignment (rotation) of the spherical system relative the Cartesian system. The net result is a

change in matrix  $\mathbf{T}$  that relates the spherical components to the Cartesian components. Consider the special case when the spherical system is rotated in the positive sense around the  $\hat{x}$ -axis through angle  $\Theta$  relative the Cartesian system. The effect on the vC-Z relations is that  $\mathbf{T}$  is replaced by  $\mathbf{T}'$ , where

$$\mathbf{T}' = \frac{1}{\sqrt{1 - (m \cos \Theta - n \sin \Theta)^2}} \quad (\text{B5})$$

$$\times \begin{pmatrix} n \cos \Theta + m \sin \Theta & -lm \cos \Theta + ln \sin \Theta \\ -l \sin \Theta & (1 - m^2) \cos \Theta + mn \sin \Theta \\ -l \cos \Theta & -mn \cos \Theta + (n^2 - 1) \sin \Theta \end{pmatrix}. \quad (\text{B6})$$

Obviously, for  $\Theta = 0$  we obtain  $\mathbf{T}' = \mathbf{T}$ , which is the matrix used in most of this paper.

At the field centre,  $l = m = 0$ , so

$$\mathbf{T}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (\text{B7})$$

for all  $\Theta \neq \pi/2$ . However, the  $x$  and  $y$  components of its first derivatives are zero only for  $\Theta = 0$ , that is

$$\left. \frac{\partial \mathcal{T}'_{ij}}{\partial l} \right|_{l=m=0} = 0, \quad \left. \frac{\partial \mathcal{T}'_{ij}}{\partial m} \right|_{l=m=0} = 0, \quad \text{for } \begin{cases} i = x, y \\ j = \theta, \phi \end{cases}$$

only for  $\Theta = 0$ . Thus, the special case we have used in this paper,  $\mathbf{T}' = \mathbf{T}$ , can be said to possess a projection ( $xy$ ) that is locally flat at the field centre. This is a reason for choosing the spherical system with  $\Theta = 0$  as a default, since all other cases would lead to a 2-D M.E. (36) with additional first-order terms that account for the coordinate system curvature within the FoV.

The matrix  $\mathbf{T}'$  can be used to adapt the vC-Z relations given in this paper, such as equation (23), to standard celestial coordinate systems such as the equatorial system or the azimuth-elevation system. In essence, assuming that the pole  $A$  is towards the Earth's North pole (in the case of equatorial coordinates) or zenith (in the case of az-el coordinates), one simply performs  $\mathbf{T} \rightarrow \mathbf{T}'$  and then interprets  $\Theta$  as either the declination (equatorial case) or the elevation (az-el case) of the centre of the FoV. Note however that the Cartesian coordinates, in which the  $uvw$  and  $lmn$  spaces are expressed, would still need to be transformed for complete agreement with these standard celestial systems, but for this common task we refer to standard texts such as Thompson et al. (2001).